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Survey of Dynamic Analysis Methods for Flight Control Design

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Nomenclature

F = total force acting on vehicle
H = vehicle angular momentum
I = moments and products of inertia
M = total torque acting on vehicle
m = vehicle mass
V = vehicle velocity vector
ω = angular momentum vector

Introduction

THE purpose of this Survey is to review technical fields that are of primary concern to the theoretical design of flight control systems, i.e., systems for the control of attitude of aerospace vehicles. Three major areas have been selected, description of vehicle dynamics, design criteria, and control theory. The literature in the first and last of these fields in quite extensive. However, the second field, design criteria, has received a very small amount of attention, despite its obvious importance. The available information is reviewed and several suggestions are offered as to directions of further work. Prediction of aerodynamic forces and moments is not considered.

In preparing this Survey, several thousand books, reports, and journal articles were reviewed. Space does not permit mention here of more than a fraction of this work, nor would so great a volume of material serve the reader well. Accordingly, a selection has been made based on the author's prejudices, but with preference for references with one or more of the following attributes, listed in no special order: 1) clarity of presentation, 2) comprehensiveness (survey articles are favored), 3) recent date, 4) ready availability (company reports are discriminated against), 5) originality, and 6) importance of the work in development of the field. Many worthwhile papers had to be omitted. An earlier work along the same lines[1] has a somewhat broader scope than the present survey and about twice as many references.

Dynamical Description of Flight Vehicles

Since the first phase of almost every flight control design effort is mathematical, it is necessary to have a mathematical description of the vehicle to be controlled. This description takes the form of a set of ordinary differential equations, in most cases. When partial differential equations are appropriate, it is common practice to approximate them by appropriate sets of ordinary differential equations as well

To represent any realistic flying vehicle completely would be a task of immense difficulty. The problem is to select the simplest approximate representation that will permit the

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flight control designer to do an adequate job. For this reason, the flight control engineer is inevitably involved in selection of the model to be used, and frequently, he must assume a considerable share of the effort of generating that model. The various idealizations commonly used are reviewed roughly in order of increasing complexity.

The Rigid Vehicle

If a flight vehicle consists of a single, unarticulated mass with no substantial amount of internal liquid, and if the vehicle is relatively inflexible, then it may be useful to idealize it as a rigid body. This was almost universal practice until about 15 years ago, when it became apparent that flexibility and fuel sloshing could not safely be ignored in designing flight control systems for some vehicles. If it seems clear that the structural and slosh frequencies are much higher than the frequencies involved in the control, then a rigid-body approach may be justified. For smaller aircraft and spacecraft, it may be an entirely adequate approximation.

General motions

The dynamical description of rigid vehicles divides into two questions, determining the forces and moments acting and determining the response to them. Since the second question is more straightforward, it will be considered first. Finding the motion of a rigid body subject to arbitrary forces and moments is one of the principal topics of classical mechanics. The fundamental physical principles are (see, e.g., Ref. 2, Chap. 4)

$$m\dot{\mathbf{V}} = \mathbf{F}; \quad \dot{\mathbf{H}} = \mathbf{M}$$

Consider a right-handed coordinate system fixed in the body, with origin at the center of mass. The rigid body has an angular velocity vector $\boldsymbol{\omega}$ and components of $\boldsymbol{\omega}$ along the body-fixed axes are ω_{xb}, ω_{yb}, and ω_{zb}. The components of the angular momentum vector $\mathbf{H}$ along these same axes are then

$$\begin{aligned} H_{xb} &= I_{xx}\omega_{xb} + I_{xy}\omega_{yb} + I_{xz}\omega_{zb} \\ H_{yb} &= I_{yz}\omega_{xb} + I_{yy}\omega_{yb} + I_{yz}\omega_{zb} \\ H_{zb} &= I_{zx}\omega_{xb} + I_{zy}\omega_{yb} + I_{zz}\omega_{zb} \end{aligned}$$

where

$$I_{xx} = \int(y^2 + z^2)\rho(x,y,z)\, dV; I_{yy} = \int(x^2 + z^2)\rho(x,y,z)\, dV$$

$$I_{zz} = \int(x^2 + y^2)\rho(x,y,z)\, dV; I_{xy} = I_{yz} = \int xy\, \rho(x,y,z)\, dV$$

and ρ is the density of the rigid body.

The derivative of any vector $\mathbf{A}$ as seen in the body-fixed coordinate system will satisfy (see e.g., Ref. 2, p. 48, or Ref. 3, p. 133)

$$\left.\frac{d\mathbf{A}}{dt}\right|_b = \frac{d\mathbf{A}}{dt} + \boldsymbol{\omega} \times \mathbf{A}$$

The first term on the right is the rate of change of $\mathbf{A}$ as seen in the external Newtonian reference frame. If the body-fixed axis system is a principal axis system (the products of inertia vanish), then the linear and angular velocities of the vehicle may be obtained from the fundamental equations

$$\begin{aligned} m\dot{V}_{xb} &= F_{xb} + m(V_{yb}\omega_{zb} - V_{zb}\omega_{yb}) \\ m\dot{V}_{yb} &= F_{yb} + m(V_{zb}\omega_{xb} - V_{xb}\omega_{zb}) \\ m\dot{V}_{zb} &= F_{zb} + m(V_{xb}\omega_{yb} - V_{yb}\omega_{xb}) \\ I_{xx}\dot{\omega}_{xb} &= M_{xb} + (I_{yy} - I_{zz})\omega_{yb}\omega_{zb} \\ I_{yy}\dot{\omega}_{yb} &= M_{yb} + (I_{zz} - I_{xx})\omega_{zb}\omega_{xb} \\ I_{zz}\dot{\omega}_{zb} &= M_{zb} + (I_{xx} - I_{yy})\omega_{xb}\omega_{yb} \end{aligned}$$

Once the forces and moments are specified as functions of time, and initial conditions are given, these equations may be integrated to obtain the velocity of the center of mass and angular velocity about the center of mass, both expressed in the body-fixed coordinate system. If this is all that is required, the solution is complete. In most problems, however, it is desirable to obtain the position of the center of mass and the orientation of the body with respect to the external Newtonian reference frame. The key to both these problems is the orientation. There are several ways to express the orientation of a rigid body. The most commonly used methods are direction cosines and Euler angles. Somewhat less common are quaternions, Euler parameters, and Cayley-Klein parameters. All these methods are discussed and compared by Robinson.[4] Whichever representation is used, there exists a set of differential equations which, together with the initial conditions and the body axis rate components ω_x, ω_y, and ω_z may be solved to give the orientation of the body. Once the orientation is specified, it is possible to develop the direction cosine matrix, even if direction cosines were not used in the integration. This direction cosine matrix may be used to transform the velocity vector from the body-fixed system to the inertial system. The inertial velocity may then be integrated directly to give inertial position, and this gives a complete description of the motion.

If the external forces and moments are particularly simple, and if the rigid body has a certain amount of symmetry, these equations can be solved in closed form. Leimanis,[5] for example, gives a review of the known cases for which this is possible. Also, if only small motions are involved, the equations may be linearized and, in many cases, solved in closed form, as will be mentioned later. In the general case, however, it is necessary to integrate these equations by either numerical or analog means.

In general, thrust and gravitational forces present little difficulty. Representation of aerodynamic forces and moments for arbitrary body attitudes is, however, a matter of some difficulty. It may be advisable to use other axis systems to express these terms. This and other matters of practical importance are discussed by Howe,[6,7] Connelly,[8] Etkin,[9] and Abzug,[10] among others.

The most common use of simulations involving unrestricted rigid-body motions has been in training simulators. These are used for training human pilots, and they must be capable of representing any motion which the pilot can command. This is the type of application envisioned by Howe[6,7] Connelly,[8] and Fifer.[11] However, they have been used for other purposes as well. The study of spin as done by Wykes et al.[12,13] and Sherer and Aguesse[14] is an interesting application. Roll resonance or roll coupling is another.[15–17]

In spacecraft studies where the atmosphere is either completely negligible or very small, large-amplitude rigid-body motions may be studied much more easily than in aerodynamic problems. It is possible, except in certain special cases, to separate the motion of the center of mass from the angular motion about the center of mass.[18,19] This cuts the size of the system of equations in half. Since the moments acting may be of rather simple form, it is possible to get closed-form solutions in a few cases.

In any event, once the moments are specified as a function of attitude, whether these are desired (control) moments or undesired (disturbance), it is possible to determine the attitude motions rather simply. A large number of studies of this sort have been undertaken in the past decade. Only a small sampling will be mentioned here.

Studies of an introductory or survey nature are those of Beletskii,[20,21] Frye,[22] Kershner and Newton,[23] Hughes,[24,25] Schett and Weil,[26] and Greensite.[27] Meteoritic impacts have been considered by Bjork,[28] Cloutier,[29] and White,[30] with the conclusion that the attitude control effects are not serious. Aerodynamic moments have been studied by DeBra,[31] Schrello,[32] Davison,[33] and Johnson.[34] Radiation pressure, either as a disturbance or as a control mechanism, has been the subject of papers by Newton,[35] Hibbard,[36] McElvain,[37] Acord and Nicklas,[38] Karymov,[39] Roberson,[40] and Harrington.[41] The interaction between satellite attitude and the earth's magnetic field has been extensively studied. (See White et al.,[42] Fischell,[43] Burrow,[44] Yu,[45] Ergin and Wheeler,[46] Gorez,[47] Chute and Walker,[48] Shevnin,[49] Renard,[50–52] Glass,[53] and Wheeler.[54]) The possible significance of gravitational torques on a rigid vehicle was pointed out by Klemperer and Baker[55] in 1957. The effect has been studied by many authors. A few of these studies are those by Robinson,[56] Frick and Garber,[57] Nidei,[58] DeBra and Delp,[59] and Pengelley.[60]

If the vehicle has an appreciable spin rate, the motion is somewhat more complicated than otherwise. Some examples of studies involving spin are those of Arkhangel'skii,[61] Thomas and Cappellari,[62] Eversman,[63] and Meirovitch and Wallace.[64] Making controlled large changes in the orientation of a spinning satellite has been considered by Cole et al.,[65] Grubin,[66,67] Wheeler,[68] Hales and Flugge-Lotz,[69] and Meyer.[70]

A great many designs of attitude control systems have been proposed for rigid space vehicles. Some are included in the references cited previously and others will be considered in later sections. Some ideas for using mass expulsion as a control mechanism are given by Adams,[71] Freed,[72] LeCompte and Bland,[73] Loebel,[74] Day,[75] Smith,[76] and Jones.[77] If mass expulsion is to be used for control, it may be necessary to modify the rotational equations of motion. (See, for example Ellis and McArthur,[78] Punga,[79] Thomson[80] and Pisacane, Guier, and Pardoe.[81]) For correlations between theoretical predictions and observed results, see Arendt,[82] and especially Pisacane and Hook.[83]

Small perturbations

Very general equations such as used in the aforementioned studies may require a sizeable computer for their solution, especially if aerodynamic forces and moments are involved. Furthermore, studying vehicle dynamics using a simulation is essentially an experimental investigation. It is necessary to make a large number of runs and to correlate the results of those runs. Indeed, analysis of the data produced may be as difficult as setting up the simulation.

Accordingly, it is desirable to look for simpler equations which have some relation to the problem. This was done quite early in the history of aerodynamics by Bryan[84] and Lanchester.[85] The essential idea is to consider only small perturbations about some steady-state equilibrium conditions so that the entire set of differential equations describing the dynamics and the forces and moments may be linearized. A linear time-invariant system usually results. Then the powerful techniques of linear systems, both for analysis and for optimization, may be used. A very large number of flight dynamics problems may be treated in this way. It is also common to use the linearized approach as a first step, even when it is intended to proceed with a nonlinear analysis later.

The first step in applying this linearization method is to select some steady equilibrium condition of flight as a basis. The most common choices in aerodynamic problems are 1) straight and level flight, 2) steady constant-altitude turns, and 3) trimmed horizontal flight with a constant rolling rate. In space applications, the steady-state might have one vehicle axis aligned along the local vertical or pointing at the sun or a fixed star. At any rate, the equilibrium is a condition at which all moments acting on the vehicle add to zero. If only small perturbations from this equilibrium condition are allowed, the dynamic and kinematic equations become linear.

If the external torques are linear as well, the vehicle can be completely described by linear differential equations, which greatly simplifies the study of its motion. Possibly the most complex examples occur in aerodynamic problems. Linearization of the aerodynamic force and moment terms gives rise to the conventional stability derivatives. In space applications, the external forces are usually functions of a small number of variables and their linearized representation may be done rather easily.

The literature on small-perturbation aerodynamic problems is quite extensive, even considering only the work slanted toward control applications. Among the more useful introductory presentations are those of Perkins and Hage,[86] McRuer, Bates, and Ashkenas,[87] Charters,[88] Babister,[89] and Blakelock.[90] The book by Seckel[91] considers a number of topics in small-perturbation analysis, both for conventional aircraft and for helicopters. He also gives aerodynamic and other data for a number of vehicles, and includes an extensive bibliography. The handbook by Kisielowski, Perlmutter, and Tang,[92] also has extensive equations and data for helicopters.

It has been shown many times that this method is capable of giving good prediction of vehicle motion in those cases where the assumptions are met (see, e.g., Heinle and McNeill[93]). There are difficulties, however, in determining the aerodynamic and inertial data which are required. Also, it is not always obvious when the rigid-body assumption is sufficiently valid to permit correct design of the flight control system. Nonetheless, it is fair to say that a major share of the understanding we have of vehicle attitude motions comes from this linearized approach. Not only can be modes of motion of a conventional aircraft be expressed in terms of stability derivatives and inertial parameters,[94] but many aspects of the behavior of spinning vehicles may be studied.[95–99]

Compound Vehicles

In recent years there has been an increasing interest in vehicles that can reasonably be represented as two or more rigid bodies that are able to exert forces on each other, either through connecting springs, cables, or bearings. Perhaps the first aircraft to require such description were jets, especially in very low-speed flight. The engine rotor and the airframe may each be represented as rigid bodies. For the parameter ranges of interest, it was found that the gyroscopic reaction forces usually could be expressed as additional torques on the airframe which are proportional to airframe rates of rotation.[12,100,101]

Many VTOL designs involve relative motion between sizable parts of the vehicle. Engine nacelles or wings may be rotated or shifted. This may cause motion of the center of gravity as well as changes in inertias. Isakson and Buning[102] derive equations of motion for vehicles of this type and also consider the problems of representing aerodynamic forces and moments when the velocity can go to zero. External loads slung from a vehicle can also give rise to problems of this type.[103]

However, the greatest number of applications of compound vehicles seems to be in spacecraft. First there are the large number of compound-vehicle schemes for passive damping of gravity-gradient satellites. The studies by Zajac,[104] Fletcher, Rongved, and Yu,[105] Etkin,[106] Tinling and Merrick[107] and Chiarappa and Nelson[108] are but a few of the many that could be mentioned. The underlying idea is to connect the bodies to each other by means of some dissipative mechanism so that the mechanical energy involved in the oscillations about the local vertical can be removed from the system. Most of the studies show that small oscillations can be damped out within an orbital period or less.

A great deal of attention has been devoted to reaction wheels, reaction spheres, and control moment gyros for either passive or active control and stabilization. Jacot and Liska[109] and DeLisle, Ogletree, and Hildebrant[110] survey the control moment gyro concept, and Liska[111] studies a high-accuracy application. Cannon[112] and DeMarinis and Huttenlocher[113] are among those considering reaction wheels. Hering and

Hufnagel,[114] among others, study the reaction sphere. Kane and Wang,[115] Likins,[116] and Cloutier[117] study the passive behavior of vehicles containing one or more rotating members.

Many unconventional types of composite vehicles have been proposed for various purposes, ranging from elementary de-spin devices[118] to masses interconnected by cables[119–121] and internal masses that are moved around for stability and control purposes.[122,123] The stability of spinning vehicles with moving internal masses is considered by Thompson and Fung,[124] with the interesting result that, for some types of motion, these masses can cause instability. Motion of spinning vehicles during slow changes of shape and inertia is considered by Sherman and Graham[125] and Gluck and Gale.[126]

Derivation of equations to represent these vehicles is apparently fraught with some difficulty. There are several examples in the literature where authors have published incorrect equations and were subsequently corrected by correspondents. More or less general approaches to deriving the equations of motion are given by Fang,[127] Harding,[128] Hooker and Margulies,[129] Pringle,[389] and Sandler.[390]

One of the more interesting recent developments is the appearance of a report by Westerwick and Brown[130] detailing a computer program that can derive equations of motion for most compound vehicles and present them to the analyst for inspection.

Effect of Vehicle Elasticity

In many aerospace vehicles of current interest, vehicle flexibility is great enough that it must be included in the model for flight control purposes. In some cases, flexibility is serious enough to be the major limitation on attitude control. Since consideration of this phenomenon is so frequently necessary, a very large amount of work has been done on it during the last decade. For general introductions to the subject, see the books by Bisplinghoff and Ashley,[131] Williams[132] or Hurty and Rubinstein.[133] Also, Braun[134] gives a review of much of the published material prior to 1964.

The dynamics of an elastic body is properly represented by partial differential equations. While there is currently some interest in control of such systems, for the most part, use of control theory requires that the controlled object be represented in terms of ordinary differential equations. To do this inevitably involves approximations of some kind, and consideration of the errors involved in these approximation is one of the principal features of the subject. There are two related approaches to this approximation problem which are commonly used. In the first, the displacement of the body from its unstrained position (this is properly a function of one, two, or three variables, depending on the number of space dimensions of the body) is expanded in terms of the characteristic functions of the partial differential equation representing the body when it is free of external forces. These are the so-called normal modes. In general, there are an infinite number of them, and consideration of only a finite number involves an approximation.

The second method of approximation involves replacing the original continuous body by a collection of rigid bodies interconnected by suitable springs. One then has to deal with a compound vehicle of the type considered previously. There is a notion of normal modes for such systems[2,3] and it is possible to adjust the parameters of the lumped-mass representation so that the characteristic frequencies agree with those of the continuous system.

The principal fields of application of flexibility considerations are for 1) large boost vehicles, 2) large aircraft, and 3) flexible space vehicles or space vehicles with flexible booms or antennas attached. In all these areas, problems have been found in which consideration of elasticity is absolutely essential to proper control design. Possibly the greatest volume of work has gone into flexible boosters. Several rather good surveys of this subject are available. See Lukens, Schmitt, and Broucek,[135] Greensite,[136,137] Ringland,[138] Smith,[139] and Wells and Mitchell.[140] Paddock,[141] Staley,[142] and Gieseke et al.[143] in a series of related reports discuss the problems of determining the lateral, longitudinal, and torsional modes. In the same series, Lukens[144] and Lukens et al.[145] discuss full-scale testing of elasticity parameters. Coupling between the various normal modes, which is a principal source of control problems, is discussed by Curtis and Hagstrom,[146] Baines and Pearson[147] and Swain,[148] among others. Bauer[149] and Stapleford et al.[150] consider both elasticity and fuel sloshing.

For aircraft problems, the fundamentals are the same, but the shapes of the significant modes are rather different because of the presence of lifting surfaces. A report prepared by J. B. Rea Inc.,[151] gives a good introduction to the process of deriving equations. The book by Argyris and Kelsey[152] gives an introduction to dynamic representation, with special emphasis on the fuselage. The report by Pearce, Johnson, and Siskind[153] contains a rather complete development. Several papers in the AGARD series on elasticity,[154] also contain introductory and survey material. Howard[155] considers specifically the effect of elasticity on control design, and Parks[156] the possible coupling of control and flutter modes. Niedenfuhr,[157] discusses the phenomena that arise when the vehicle has an appreciable rolling rate. Pearce,[158] considers the problem of truncating the elastic representation as well as possible with a finite number of modes. Mode coupling is considered by Pass and Pearce.[159,160] Wykes and Mori[161] present data on some specific aircraft and analyze the control problems, using elementary control theory. Techniques for control of flexible vehicles are reviewed by Greensite,[162] Cunningham and Higgins,[163] and Rynaski, Whitbeck, and Dolbin.[164]

Since most extra-atmospheric spacecraft have been relatively small and rigid, a somewhat smaller number of elastic studies have appeared for such vehicles. Some interest has been shown, however, in very large and flexible spacecraft[165,166] and spinning elastic space stations.[391–393] The most popular subject has been gravity-gradient satellites with flexible members. Some of the more recent studies are those of Liu and Mitchell,[167] Etkin and Hughes,[168] Modi and Brereton,[169] Florio and Hobbs,[170] and Newton and Farrell.[171]

The general effect of considering elasticity is to add to the system of equations one coupled second-order differential equation for each mode of motion considered, or each additional point-mass included in the model. It is not always clear just how many modes should be included, but the mode frequencies and their relationships to the desired control frequencies give some guidance.

Fuel Sloshing

Many flight vehicles have an appreciable fraction of their mass in the form of liquid fuel and/or oxidizer. If these liquids are stored in sizable tanks, the motion of the fluid within the tank may come to have a decisive influence on the motion of the vehicle as a whole. This phenomenon is most serious in large liquid-fuel boost vehicles, though problems have also been observed in aircraft.

The motion of a fluid is properly described in terms of partial differential equations. As in the case of elasticity, however, it is possible to represent the motion in terms of normal modes and achieve a reasonable description with a small number of modes. Each normal mode may be represented by an equivalent spring-mass-damper system. Studies of this means of representation are surveyed by Cooper,[172] Abramson,[173] and Fontenot.[174]

The fundamental studies in the field are those of Kachigan,[175] Miles,[176] Schmitt,[177,178] and Bauer.[179] More recently, there has been interest in flexible tanks,[180–182] longitudinal sloshing,[183–185] and nonlinear representations.[186,187] Some of the references cited in the previous section consider fuel sloshing as well as elasticity, e.g., Refs. 136–138 and 140. The effect of fuel motion on the control design problem is rather similar to that of elasticity.

System Identification from Dynamic Response

In arriving at a dynamical description of the vehicle, it is necessary to decide how complex the model must be (rigid or flexible, including slosh or not, etc.). Furthermore, it is necessary to estimate the parameters that appear in the selected model, such as inertias, aerodynamic stability derivatives, stiffness distributions, etc. These estimations are subject to a number of uncertainties, and it is desirable in many cases to have the capability of verification of these data in actual flight.

This problem becomes the concern of the control designer in two ways. Most obviously, the flight control design depends on the data used. If these data are in error, that fact should be ascertained so that the design may be corrected. Furthermore, it may be that some sort of adaptive control scheme is desirable, and for this purpose, some level of real-time system identification is required.

The first efforts in this field were concerned with nonparametric determination of vehicle dynamics, that is, determination of the dynamic behavior without any particular assumption as to the form of the dynamical equations other than assuming them linear. For example, frequency response was measured directly by oscillating control surfaces sinusoidally and measuring the resulting sinusoidal oscillations in body rates, etc. This process was so time-consuming that generally the weight of the vehicle changed substantially during the process by use of fuel. Furthermore, it is completely inappropriate if the vehicle has no steady-state flight condition, as, for example, a boost vehicle. Next, transient inputs were used, and the transfer operator of the airframe was determined by taking numerical Fourier transforms of the input and output signals. While this cut the flight time drastically and made steady-state conditions much less important, it was found that numerical Fourier transformation was very demanding as to accuracy of the data. Furthermore, the system transfer operation is an amalgam of many separate pieces of data. If the transfer operator is different from that expected, it is not possible to say in which part of the model the error originates. Since it has been shown[188] that these nonparametric methods take substantial amounts of measurement to obtain good accuracy in the presence of noise, there has been limited recent interest in this approach.

The next step was to transfer to a parametric representation. That is, some assumption was made as to the form of the equations, and it was necessary only to estimate the values of certain coefficients in this model. One of the first methods of this type was proposed by Gowin[189] and also applied by Howard.[190] All the state variables and their derivatives were measured, so that it was possible to substitute all variables into the system equations of motion. If the equations are not satisfied at every instant, the equation coefficients must be adjusted. By taking data for a large number of time points, and adjusting the coefficients by a least-squares process, the error in satisfying the differential equations can be minimized. The more time points used, the smaller will be the effect of noise in the measurements. This method does not require the use of any special inputs such as sine waves, and in many cases, normal maneuvering and control inputs are adequate. The principal difficulty is the large amount of data required, including the time derivatives of all state variables. Some of these may be difficult to measure and record accurately.

A method that circumvents these particular difficulties was proposed by Valstar[191] and by Young.[192] They use a string of low-pass filters on the (single) input and (single) output. The outputs of the filters are then used to estimate the coefficients of the numerator and denominator of the transfer operator of the unknown time-invariant linear system.

Discrete-time versions of the equation-matching procedure are given by Liff[193] and Smith,[194] based on ideas drawn from linear sampled-data systems. Real-time recursive regression methods are proposed by Marchetti[195] and Lion[196] in which the estimate of coefficients is improved at each time instant until they reach a steady state, which is the true value. Closely related methods are given by Yore and Takahashi[197] and by Hoberock and Kohr.[198]

It appears that these methods are definitely practical. Several of them work in real time, providing a good estimate of system parameters after a certain period of operation. For some methods, global convergence can be proved and the system can be identified with zero error in finite time if noise is not present. They should be of substantial benefit to flight control designers.

Performance Criteria

Once the controlled system has been described, the next task is to decide what the control is to accomplish. Of all the subjects covered in this survey, this question of criteria is the one in the most unsatisfactory state. Control theorists and practitioners have discussed this subject for years, and have arrived at no particular result. A number of years ago Graham and Lathrop[199] considered a number of criteria, based on step-function response in linear time-invariant systems. Walkovitch et al.[200] reviewed the same subject more recently and Magdaleno and Wolkovitch[201] reviewed the corresponding problem for random inputs. Also, Schultz and Rideout[202] reviewed the entire subject in 1961. In looking at all this work, the subjective aspect of criterion selection is quite evident. There is a considerable tendency to fit the criterion to the analysis method to be used rather than to the requirements of the problem. This indicates that, in most cases, the real technical requirements are not explicitly understood.

It is universal practice to define performance criteria in terms of dynamic response alone, e.g., rise time, peak overshoot, damping ratio, M_p, etc. Certainly it is essential to obtain a satisfactory dynamic response with the controlled system, but this is only part of the design problem. Ignoring other factors is a source of much confusion about the criterion question. The criterion should be broadened to include all the things with which a control designer must be concerned besides dynamic response, such as hardware cost, weight, reliability, maintainability, availability, commonality with other systems, man-hours required in design and test, etc. At present, these factors enter more or less subjectively into the designer's decision process. They should be rendered explicit, early in the process if a genuinely optimal system is to be evolved.

Since there are a number of aspects to be considered, a vector criterion might be suggested. One numerical value would be attached to each of the foregoing considerations. There has been some interest recently in vector criteria, but this is really begging the question. Some single system ultimately will be built. This, is, in effect, a scalar decision. The resulting system is felt to be "better," whatever that means, that all other candidates. Human designers arrive at this sort of decision. It is not unreasonable to expect that a suitable scalar criterion could permit a computer to do as well.

Current design practice is to postulate a system, try it out in a number of ways and see if it seems satisfactory in all aspects. If not, some change is made and the process is repeated, until the design is satisfactory in all respects. At no time in the process is the designer forced to state explicitly what criteria he is using and how he is weighing the different aspects of the design. The result of this process is clearly nonunique. It depends to some degree on luck and on the experience, not to say prejudices, of the designer.

If an objective criterion could be established, expressing the true goals of the design effort, it seems possible that, in view of the current status of theory and computation, an optimum system could be synthesized entirely by a computer program. The savings in engineering man-hours alone might justify the effort of setting up such procedures. There is, however, another reason for interest in this approach.

It seems quite probable that at some time in the future, entire aerospace systems will be designed optimally by a computer program. Already progress is evident on many fronts, i.e., automatic optimum electronic circuits, optimal trajectories, optimal guidance hardware, optimal structure, and even optimal automatic airframe design.[203] System design is now a long and expensive iterative process. If it can be automated, the savings in both time and money may be very great. The flight control designer must be prepared to make his contribution to this effort.

This, however, lies somewhat in the future. There are other aspects of the criterion question which are of more immediate interest. One relates to the problem of the variability of the dynamics of the controlled system. Most aerodynamic vehicles undergo sizable changes in dynamical characteristics over their flight regimes. This fact has stimulated interest in both gain-scheduling and adaptive control. There are no accepted criteria for evaluating how well control systems perform in this context. Furthermore, it is not at all clear that adaptive or gain-scheduled control, with their attendant complexities are even needed. The results of Horowitz,[204] Krachmalnick, Vetsch, and Wendl,[205] and Dyer, Noton, and Rutherford[206] indicate that the "best" of all fixed controllers may be very "good" indeed.

One approach to a criterion in this type of problem is drawn from the theory of games. The flight envelope of a vehicle may be viewed as some closed region X in a suitable parameter space. Each point $x \epsilon X$ in the space defines a single flight condition. Now assume that some dynamic response criterion $Q(x)$ can be defined at each $x \epsilon X$. As an example, the integral-square error in response to a step input might be used. Then, it would be reasonable to attempt to make the largest value of $Q(x)$ (over-all $x \epsilon X$) as small as possible. The controller should be selected, then, to minimize $\max Q(x)$, where x ranges over X. This can be viewed as an adversary situation where one player selects the flight condition x in such a way as to maximize $Q(x)$, while the other player selects the controller in such a way as to minimize $Q(x)$. A considerable body of theory exists for games of this type,[207,208] and, in fact, some applications have been made along these lines.[209,210]

As to the criterion $Q(x)$ to be used at each flight condition, it seems possible that most of the criteria used in general control work will not be appropriate. They generally suggest that the response should be as fast and well-damped as possible. In aerospace applications there are typically limitations on the speed of response desired. Many of the more successful recent designs have used some form of model-matching. They have attempted to make the vehicle respond like some preselected "ideal" vehicle, some guidance can be obtained in the aircraft case from handling qualities criteria,[211,212] which are based on pilot opinion. In the case of missiles, there is no such ready source of models, but the approach should still be beneficial.

There is another use of minimax ideas in control which has received some attention. Suppose that at some specific flight condition, one considers the error in response to a step input. If it is desired to minimize the *maximum value* of the error during the response, then some of the same ideas may be applied. This type of application has been considered by Johnson,[213] Harvey,[214] and others.

Optimal Control

In studying control problems, there are two general approaches which may be taken. The first, and most traditional is one of postulating a system and analyzing its behavior. Changes are made if the behavior is not proper, and the process is repeated. The fundamental analytical tools are those of analysis, i.e., finding the behavior of a given system. The second possible approach is that of direct synthesis. An objective is established and an attempt is made to find that controller which best meets the objective, subject to appropriate restrictions.

The many tools available for analysis in control problems will not be considered here. They are, for the most part, reasonably well-known and the reader may consult[1] for a survey of these techniques. Here will be considered only methods for synthesis of control systems to optimize certain criteria.

The literature of optimal control is now quite extensive. Many problems have been solved. Not all of this work is equally applicable to flight control design problems, however. In reviewing this subject, primary emphasis will be placed on those aspects which appear to be a greatest benefit in the design of aerospace attitude control systems, and a number of applications will be mentioned.

As a basis for discussion, we consider a rather general deterministic continuous optimization problem. This is the well-known Bolza problem, which has been one of the fundamental problems in optimal control and the calculus of variations. Consider a system defined by a set of differential equations $\dot{x} = F(t,x,u)$, $t_1 \leq t \leq t_2$, where x is an n vector, F is an n vector function defined on some suitable region of its arguments and u is an m vector of controls which are, to some degree, at the disposal of the designer to influence the behavior of the system. t is the independent variable (time) and the dot indicates differentiation with respect to t. Suppose that the boundary conditions are subject to the equations $R[t_1,x(t_1), t_2,x(t_2)] = 0$ where R is an s vector function defined on some suitable region. Suppose further that there are certain inequalities to be satisfied at all points: $Q(t,x,u) \leq 0$: $t_1 \leq t \leq t_2$ where Q is an l vector function.

To define a trajectory it is necessary to specify t_1,t_2, and the functions $x(t)$ and $u(t)$ for all $t_1 \leq t \leq t_2$. Now define K as the set of all trajectories that satisfy the differential equations, boundary conditions, and inequalities. Any trajectory $\Gamma \epsilon K$ is called an "admissible" trajectory. Suppose two functions f and g are given which define a criterion function J;

$$J = g[t_1,x(t_1),t_2,x(t_2)] + \int_{t_1}^{t_2} f[t,x(t),u(t)]\,dt$$

Then, assuming the functions involved are sufficiently regular once an admissible trajectory is selected, the value of J for that trajectory may be computed. The value of J naturally depends on which trajectory is chosen.

The optimum control problem may now be stated. It is to find a trajectory Γ^* such that

$$J(\Gamma^*) = \min J(\Gamma); \qquad \Gamma \epsilon K$$

In other words, we want to find the (admissible) trajectory which makes J a minimum.

Nothing has been said up to this point about the regularity of the various functions appearing in this formulation. To begin with, assume that all functions have three continuous derivatives with respect to all arguments for all values of the arguments. Also assume that the matrix $\partial Q/\partial u$ has maximum rank everywhere, and that $u(t)$ is at least piecewise continuous. Then we have approximately the traditional Bolza problem of the calculus of variations.[215]

Many other problems have been considered. Some are more general than the one given here; others are less. Before reviewing more specific studies, it may be of interest to mention some of the references which have a broad coverage, i.e., books and survey papers.

Considering the amount of current research interest in optimal control, there are not many books available. The classic book by Bliss[215] has been in print for over 20 years, but there was relatively little application of variational ideas to control problems until interest in the subject was stimulated by the work of Pontryagin and others in the USSR in the late 1950's. The Pontryagin group published a summary of their results which has been translated.[216]

In 1957, Newton, Gould, and Kaiser[217] published a book on what we would now call linear time-invariant systems with quadratic criteria and stochastic inputs. The methods employed are related to those of Weiner filtering and its extensions. They did not, of course, have available Kalman's reformulation of that problem which renders the solution much more accessible. Within that limitation, however, the book is well-written and was doubtless influential in causing control engineers to think in terms of optimal systems.

Chang's book[218] (1961) is somewhat along the same lines, though he does include some mention of the Pontryagin maximum principle and of dynamic programming. His presentation of optimal control theory in the current sense of the term is not very complete. Kipiniak[219] discusses a number of questions related to the computational problem. In addition, he presents some ideas on necessary conditions and stochastic problems. Though plagued with notational difficulties, this is a very interesting piece of work, especially considering the time at which it was published. Pars' book[220] on the calculus of variations has a number of interesting examples and a good deal of illuminating discussion, but it is very classical in approach, and makes no contact with control applications.

The books by Bellman[221,222] and Bellman and Dreyfus[223] present the dynamic programming approach and some discussion of other ideas. Tou[224] gives an introduction to several aspects of optimal control and gives a more thorough treatment[225] to the discrete-time problem. Fel'dbaum[226] has a well-written introductory volume with a good selection of material.

Athans and Falb[227] have produced a thorough and comprehensive book on optimal control. Their selection of material is very much in line with current research emphasis. Aoki[228] and Lee[229] have published books dealing with important problems in stochastic optimal control, and presenting many useful results. Several books have appeared which consist of collections of individual papers. Such papers are, in many cases, of broader interest than the usual journal article. See the collections of Bellman,[230] Leitmann,[231] Balakrishnan and Neustadt[232] and Leondes.[233] Some literature surveys are also available. Bellman published a bibliography of dynamic programming.[234] Fuller[394] published one on the maximum principle of Pontryagin. General surveys have been done by Conti,[235] Oldenburger,[236] Paiewonsky,[237] and Athans.[238] The reader will probably find the last two the most useful.

Necessary Conditions for Deterministic Problems

A major share of the effort which has been expended on optimal control problems has gone into the development of necessary conditions. They have now been derived for a large number of cases. Such conditions serve two purposes. First, they frequently make possible deductions about the structure of the optimal trajectory or the optimal control. Second, it may be possible to use them to find the actual optimum.

Basic necessary conditions

The use of these necessary conditions permits the solution of many optimal control problems. When all the necessary conditions are applied, the result is a two-point boundary-value problem whose solution (if it can be found) is at least a candidate for the optimal trajectory.

The fundamental results for continuous lumped-parameter systems are given, via the Pontryagin approach in.[216] The problem considered there is not quite so general in some ways as the one stated at the beginning of this chapter, but the extension is not difficult. Berkovitz[239] and Hestenes[240] show the relationship between the results of the Pontryagin group and the classical calculus of variations. Berkovitz also shows the contact between the classical calculus of variations and the principle of optimality of Bellman. Chang[241] and Neustadt[242] present treatments of more general problems.

Necessary conditions for discrete-time systems have been derived using several different approaches. See the papers by Jackson and Horn,[243] Halkin et al.,[244] Pearson and Srindhar,[245] Kleinman and Athans,[246] and Holtzman.[247] Several of these papers use ideas derived from nonlinear programming, and others use a geometric approach similar to that of Pontryagin. Dreyfus[248] uses dynamic programming.

State space constraints

If the matrix $\partial Q/\partial u$ fails to have maximum rank (this can certainly happen if the Q's are independent of some component of u) then the usual derivation of the necessary conditions requires some modification. Pontryagin et al. (Ref. 216, Chap. 6) were apparently the first to consider a version of this problem, and their approach has been applied (see Ref. 249). Berkovitz[250] considered a somewhat more general problem, from a different point of view, and later[251] reviewed and correlated the available results. Guinn's treatment[252] is in somewhat the same spirit. Warga[253] also obtained necessary conditions using yet a different method of argument. For additional treatments, see Refs. 254, 255.

By various devices, the state space constraint problem may be transformed into a standard minimization problem at the cost of some increased complexity. Penalty functions have been considered by Russell[256] and Okamura[257] and a somewhat related idea using an isoperimetric constraint has been analyzed by Chyung,[258] following an idea of Lee.[259] The most popular application has been to systems with rate-limited actuators.[260,261]

Singular controls

In many optimal control problems, the usual necessary conditions are not sufficient to specify uniquely the control and the trajectory. The most common instance is the case where there is a range of controls which will give the Hamiltonian its minimum value, or under some circumstances the Hamiltonian becomes independent of the control. The optimal trajectory may still be unique, but application of the necessary conditions is ambiguous. Johnson[262] gives an introductory survey of problems of this type.

The use of the second variation as proposed by Kelley[263] seems to be the most successful approach to the problem.[264,265] Haynes suggests using Green's theorem.[266] See also the investigations of Refs. 267–272.

Minimax problems

If the differential equations describing the control problem are $\dot{x} = F(t,x,u,v)$ where $u(t)$ and $v(t)$ are two control vector functions (not necessarily of the same dimension) and if the objective of the control u is to minimize J as in the original problem, while the objective of v is to *maximize* J, the problem is called a differential game.

The fundamental studies of this problem are those of Berkovitz,[208] Warga[253] and the book by Isaacs.[207] Ho,[275] in reviewing Ref. 207, gives a good short summary of the subject, and helps to establish the correlation between Isaacs' work and the more traditional language of control problems. Other reviews of the subject are given by Ho,[276] Ho, Bryson, and Baron,[277] and Johnson.[278]

The problems of assuring that the output does not deviate by more than a prescribed amount is considered in Refs. 275 and 280. The situation where one adversary selects the initial conditions and the other selects control parameters has been studied.[281,282] Some additional remarks on the minimax approach and its relationship to the sensitivity or adaptation problem were given in the section on performance criteria.

Miscellaneous Extensions

Several of the other conditions of the fundamental problem have also been weakened. It was assumed that the control $u(t)$ was piecewise continuous. Necessary conditions can be given in the cases where it is not piecewise continuous because of either chattering (sliding state),[283–285] or because the control can go to infinity.[286–291]

Discontinuities in the state variable $x(t)$ is not allowed in the standard theory. Problems with such discontinuities are considered in Refs. 292–295. Mass change when a booster stage is separated is an example of such a discontinuous state variable.

If the functions $F(t,x,u)$ are themselves discontinuous, some modifications are required.[296,297] If time lags are present, so that the system is described by differential-difference equations rather than differential equations, then necessary conditions are considered in Refs. 298 and 299.

Existence and Uniqueness of Optimal Trajectories

Before attempting to find an optimal trajectory, it is of some interest to know whether it exists and, if so, whether it is unique. It is perhaps not enough to argue the existence of an optimal control on the grounds of physical intuition concerning the system being considered. The physical system might indeed have an optimal control, whereas the mathematical model proposed for the system might not.

Despite considerable advances in the sufficiency theory in the last three or four years, it is not possible to give sufficient conditions for problems that are as general as those for which necessary conditions can be given. Many results are, however, available. The subject is reviewed by Athans,[238] Cesari,[300] and Schmaedeke.[301]

Under some conditions, it can be shown that a trajectory satisfying the standard necessary conditions must be the optimum.[302] If one trajectory exists meeting all the required boundary conditions, there are additional requirements to assure that an optimum trajectory exists.[303] It is also possible to give conditions under which there will exist any trajectory meeting the boundary conditions.[304] Under certain circumstances, the piecewise continuous nature of the optimal control may be assured.[305] The most general existence theorems currently available are those of Mangasarian[306] and Cesari.[300]

Computation of Optimal Trajectories

Since only a small number of interesting optimal trajectory problems may be solved in closed form, it is necessary to consider the problem of computing optimal controls on an analog or digital computer. There are two general approaches one can take to this task, direct methods and indirect methods.

The direct methods do not make use of the necessary conditions of the calculus of variations or the maximum principle. They minimize the cost functional directly by considering various changes in the control or the trajectory itself. Indirect methods are based on the necessary conditions, and always involve a two-point boundary-value problem or its equivalent.

Direct methods

Various direct methods are among the more successful computational approaches. The best-known of these is the Bryson-Kelley gradient method,[307,308] which is reviewed by Kelley.[309] The technique has been extended in a number of directions and applied to many problems. One of the first extensions was to the case of bounded controls.[310] It has also been modified to include state variable constraints[311–314] by various means. State equations with discontinuous right-hand sides have been considered in Ref. 315.

The major difficulty with the method is its slow convergence near the optimum. Bonner[311] and Rosenbaum[316] discuss the problem of accelerating the convergence in complex problems. A somewhat more effective way of accelerating convergence is through use of the second variation.[317–319] Gottlieb[320] suggests use of one of the necessary conditions to accelerate convergence. Noton, Dyer, and Markland[321] suggest a scheme in which, at each step of the iteration, the change in control δu is selected by means of complete solution of an analytical control design problem. In the example given, convergence is extremely rapid, but the computation involved is lengthy, even by gradient method standards.

If there is a standard method for computing optimal trajectories, the gradient method is it. It converges slowly but rather reliably. Experience shows that it will converge even from extremely poor starting conditions. Convergence slows down, however, as the optimum is approached. At the cost of some additional effort, this problem, too, can be overcome. The method does usually require, however, a considerable amount of experimentation with various computational parameters before it is successful.

Most of the other direct methods involve reducing the original continuous problem to a discrete-time problem of some sort involving only a finite number of variables,[322–325] after which it may be solved using methods for ordinary minimization which are discussed in the above references and also in Refs. 326–329.

Indirect methods

Probably the most obvious way of solving the two-point boundary-value problem is to convert it into a sequence of initial value problems. The unknown initial conditions are guessed and the equations are integrated. In general the resulting final conditions do not match. It is then necessary to change the guesses of the initial conditions in such a way that the final conditions will be met. This may be done either by a computed adjustment[330] or even manual adjustment.[331] A more sophisticated method for making this adjustment was proposed by Neustadt[332] and extended by others.[333] Another approach to the two-point boundary-value problem is that of Goodman and Lance,[334] of which several subsequent variations have appeared.[395,396]

The method of McGill and Kenneth[335] is another technique which has been relatively successful. Several variations and applications have appeared Refs. 336 and 337.

All these methods involve integration of sets of differential equations whose stability properties vary from the undesirable to the impossible. The equations are always unstable to some degree (see Ref. 219, p. 42). Because of this instability, errors will tend to propagate on either analog or digital computers. Jordan and Shelley[338] propose an idea for improving this situation through a change of variables. In several of the computing techniques mentioned previously, use is made of penalty functions. The validity of this device is considered by Fiacco.[339] Several computing methods are reviewed and compared by Kopp and Moyer[340] and by Isaacs.[341] Few investigators use more than one or two methods and work done with different programmers on different machines at different times is hard to compare. The collection of Balakrishnan and Neustadt[232] contains a variety of methods.

Optimal Controllers

In the work cited up to this point, primary interest has been in the optimal trajectory. The solution to the problem was a control $u^*(t)$, $t_1 \leq t \leq t_2$, which minimized the performance criterion J. Interesting though this is, it is not of much direct assistance to the designer of a flight control system. He is faced with the problem of designing a device of some kind which will accept inputs from various sensors and actuate the control mechanisms. He is concerned with controllers, rather than with specific control time histories.

Conceptually, the gap could be bridged as follows. Modify the optimum control problem in such a way that the initial conditions on all state variables are given, i.e., $x(t_1) = c$, where c is a known vector. The optimal control u^* will depend, of course, on what these initial conditions are, i.e., $u^* = u^*(t,c)$. Now suppose that, using one of the methods of the preceding section, it is possible to calculate $u^*(t,c)$ for any t and c desired. Then an optimal controller could be constructed. It would be necessary to measure all the state variables. This would establish c. Then $u^*(t,c)$ could be calculated. Once this was available, it would be necessary only to evaluate it at t_1, i.e., $u^*(t_1,c)$ in order to determine what the controller should be doing at the moment. This process would have to go on continuously as x is changing. Of course, t_1 would be changed as time proceeds.

Awkward though it is, this is a solution to the optimal controller problem. It gives a rule by which one can decide what the control should be doing at each instant, based on what the system is doing at the moment. This is what is meant by a controller.

There are several objections to this procedure, besides its rather extreme complexity. In the first place, it seems inefficient to evaluate $u^*(t\,x)$ $t_1 \leq t \leq t_2$, when all that is really needed is $u^*(t_1,x)$. Kalman[230] suggests a method for getting at the control law directly, using the Hamilton-Jacobi equation. The idea is extended by Bridgeland[342,343]. It is used by Wonham[344] in a simple problem to derive the optimal controller. There are considerable practical difficulties in solving this equation, but it represents an approach.

Perhaps a more serious objection to the type of controller proposed previously is that it requires a knowledge of all the state variables. For some vehicles, this condition might be met, but for many others there is some question as to how many state variables should be considered, and certainly not all of them will be available. Indeed, in the conventional autopilot problem only a rather small fraction of the state variables are measured.

At present, there is no general approach available for overcoming this difficulty. It seems inevitable that if state variables are missing, a stochastic approach must be employed. More will be said about this in a later section.

One controller problem that can be solved rather completely is the linear system with a quadratic cost function. If the system is described by equations of the form

$$\dot{x} = A(t)x + B(t)u$$

and the function to be minimized is of the form

$$J = x'(t_2)Cx(t_2) + \int_{t_1}^{t_2} [x'(t)D(t)x(t) + u'(t)E(t)u(t)]\,dt$$

where C, D, and E are positive definite or semidefinite matrices and x' is the transpose of x, then it is possible to determine the optimal controller $u^*(t,x)$ and it turns out to be linear in x. A number of problems along this line have been solved. The books by Merriam,[345] Tou,[224,225] and Athans and Falb[227] all contain treatments of this theory either in the continuous or sampled-data case. In the Russian literature, this is known as the analytical design of controllers, and it has been treated in a number of versions, including some nonlinear extensions. (See, for example, Refs. 346–348.)

Optimal controllers are derived by a variety of special techniques in Refs. 349–355. The surveys of Letov[356] and Eaton[357] give further references.

Optimal Stochastic Control

It the control problem has some essential stochastic element, then the method of treatment must be somewhat different. The most common such elements are random inputs, noisy sensor data, and noisy or uncertain system parameters. These obviously can occur in a number of combinations, and in linear or nonlinear systems. For the most part, the theory of these problems is not nearly so far advanced as it is for deterministic ones.

Wonham[358] gives a survey of the subject. The book by Lee[229] also gives an explanation of many of the fundamental ideas. Aoki's book,[228] although restricted to discrete-time systems, gives an excellent treatment. Not many problems of realistic complexity have been solved, but this is a field of essential importance. In 1963, Rosenbrock[359] formulated a very general stochastic control problem, and solved it for a simple example. The results he obtained were very interesting. He was able to deduce the optimal controller for a linear system as a function of the level of the noise in various parts of the control loop. He found that if the uncertainty in the state and input of the system were very low, the optimal controller tended to be of the bang-bang type (the control displacement was limited) as has so frequently been found in deterministic problems. As uncertainty in the system increased, the optimal controller tended to become linear. This raises the conjecture that perhaps the real reason for the wide use of linear controls in the past has to do with the existence of uncertainty in controlled system, rather than the ease of analysis of linear systems, as has so often been alleged.

At any rate, it seems that many real-life control problems, when properly phrased, will have stochastic elements. Further developments in theory and computation should make it possible to find the true optimal controllers.

One of the few general results in this field is the decomposition theorem[360,361] which states that, for continuous linear time-invariant systems, with quadratic loss functions, if some state variables cannot be measured, the optimal controller, consists of an optimal estimator for the missing state variables, followed by an optimal controller designed on the basis that all state variables are available. The problem of optimal filtering itself, in the linear case, was reformulated by Kalman and Bucy[362,363] and solved in a form much more tractable in control work than that of Wiener.

A number of particular studies have been done, of which only a few will be mentioned, e.g., Refs. 364–369. One study by Johansen[370] is interesting in that it attacks directly the problem of limiting the complexity of the controller, a matter of considerable practical concern.

Sensitivity and Adaptive Control

Many aerospace vehicles have dynamical characteristics that are either unknown or highly variable, or both. The control designer must take this into account in order to achieve satisfactory results. Three ways of approaching the problem have been found useful. On one hand, it is possible to study the effect of these unknown changes on system performance and to try to design the (fixed) controller so that these effects are tolerable. This is the sensitivity approach. On the other hand, it is possible to make continuous measurements of system behavior and determine the dynamical characteristics. The controller parameters can then be adjusted based on these measured values of the system parameters. This is the adaptive approach.

Finally, if the system dynamics are variable, but in some deterministic way, controller parameters may be made to depend on flight condition in such a way that satisfactory performance is achieved at each flight condition. This is the gain-scheduling method.

All three of these approaches have been used and/or studied. The gain-scheduling method will not be considered here, as it is standard. The other two will be reviewed separately, but it is essential to keep in mind that they are really directed at the same problem. Their relationship was discussed in an earlier section.

Sensitivity

Originally, the study of sensitivity was concerned only with the analysis problem. That is, given a dynamical system, how does the response vary as some of the system parameters change? The book by Tomovic[371] and the survey of Kokotovic and Rutman[372] review this aspect of the theory.

More recently, however, there has been more concern with the active control of sensitivity, i.e., keeping it below some desired limit. Horowitz[373] developed a frequency domain which permits control of sensitivity by a cut-and-try process. It is not an optimal procedure, but Horowitz obtained some remarkable results with it.[204]

In linear problems, adjoint variables may be defined which may be interpreted as sensitivity measures. These variables satisfy the adjoint differential equations. If they are added to the original variables, the results are a set of quantities which contains information on both the system state and the sensitivity of system state. The augmented state variable may be used in an optimization problem such that the sensitivity is controlled or actually minimized. Kreindler[374] formulates several problems of this kind and reviews other similar studies. Kahne[375] takes a somewhat similar approach. Johnson[376] considers a problem in which the fundamental system equations are nonlinear, but sensitivities are defined in terms of small perturbations from some (nonlinear) trajectory. The problem is to find that trajectory for which the sensitivity is a minimum. Several recent studies relate to the question of how to design feedback systems so as to get satisfactory sensitivity.[377–380]

Adaptive Control

The book by Mishkin and Braun[381] reviews the earlier work in the field and there are a number of surveys. Some of the more recent are Refs. 382–385. The problem of system identification has already been considered. This forms the basis for many of the adaptive schemes that have been proposed.

Some of the more recent approaches make use of a model of the controlled system in some way. That is to say, internal to the controller is a simulated model of the system being controlled. Parameters of this model are updated in some logical fashion, based on the difference between the response of the model and the response of the actual controlled system. The studies of Whitaker[386] and Hiza and Li[387] are among many that might be mentioned. Additional references are given in the surveys cited previously. These model reference systems give excellent performance. They adapt very rapidly and the resulting control is good. Their principal shortcoming is controller complexity. As stated elsewhere, this should be considered from the outset in the over-all controller design problem. In adaptive control studies, however, the objective is to optimize the dynamic performance alone. In this, considerable success has been achieved.

In more recent work on the adaptive problem (e.g., the books by Sworder[388] and Aoki[228]), the theory is becoming indistinguishable from that of other branches of control work. The correct approach is to include the variability of the plant parameters in the design problem at the outset, and derive an optimal over-all control strategy. Whether this turns out to be an adaptive solution or not may be a question of semantics.

Conclusions

It is possible to derive equations of motion taking into account all important dynamical effects. Just how many effects need to be included for satisfactory control design, is perhaps a matter of judgment, but, if desired, it is always possible to be conservative by including representation of additional effects. The dangers of various approximations seem to be relatively well understood. On the whole, then, dynamical representation, though sometimes tedious, is not a limitation.

The criteria currently in use for control system design are far from complete. They consider only the dynamic performance of the closed-loop system and do not account for numerous other factors that influence control system design. The principal such factor is controller complexity. Satisfactory designs are now achieved by imbedding the dynamic analysis study in a larger, manual, decision-making process. Efforts should be made to quantify all relevant criteria and put them into the design process at the outset. Then perhaps a controller that is truly optimal could be derived by a purely automatic process.

If such a criterion were available today, mathematical optimization might not be possible. However, perhaps modest extensions would make it possible, especially in the computational field. Although it is possible to determine optimal control time histories rather readily, the design of optimal controllers is not now possible except in special cases. It seems quite likely that when control design problems are properly phrased, there will be essential stochastic elements; so the field of stochastic optimal control is the one that presents the most promise for solving the problems designers face.

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Concept and Characteristics of the Concorde Exhaust Noise Suppressor

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Performance and design considerations for the Concorde exhaust noise suppressor are discussed. The suppressor makes a radial injection of mixing air in the primary jet stream using ten lobes in the form of triangular prisms hinged to the divergent section of the ejector nozzle. When suppression is no longer necessary, a feedback linkage allows the lobes to retract so as to eliminate all thrust losses in the cruise position. Results of model tests used to develop detailed geometry of the design are also presented. Performance of a retractable suppressor is compared with one of fixed geometry taking into consideration the effects of weight, fuel consumption, and the initial climb trajectory of the aircraft. It is concluded that the Concorde suppressor provides substantial noise attenuation at low weight and without cruise thrust loss.

Introduction

ENGINE noise during takeoff and landing has been a problem ever since turbojet aircraft were first introduced to the airlines. Exhaust noise, a major factor at this time, led to development of turbojet-noise suppressors and also contributed later the introduction of dual flow (turbofan) engines having reduced exhaust noise. Noise reduction efforts were then oriented towards attenuation of acoustical emission from the fan and compressor. The situation has changed since SST studies began and exhaust noise of the required high-thrust, low frontal area engines is again a major factor.

Engines of this type inherently have high exhaust speeds which are sources of acoustic emissions of high intensity (the acoustic energy of a jet being proportional to the eighth power of its relative velocity). At takeoff, exhaust noise of this type engine is much greater than compressor noise; it is only in approach flight with reduced thrust that acoustic emission from vanes and blades of the engine (compressor and turbine) plays an important part in the noise level of an SST. Although the high jet velocity generates noise, it offers in return a high thrust which allows the aircraft to climb rapidly, thereby reducing the zone around the airport exposed to intense noise. Another favorable characteristic is that the divergent portion of the nozzle (necessary for high thrust at supersonic speeds) slightly reduces acoustic emission from the engine primary nozzle at takeoff conditions. Furthermore, the horizontal diffusion of noise from multi-engine aircraft is reduced by engine grouping which causes noise from inboard jets to be masked by outboard jets. In the Concorde, this mask effect may be reinforced by interaction between the jets (closely spaced) of each pair of engines. In spite of these attenuating influences, the noise level of every large supersonic aircraft remains high, and the incorporation of an exhaust noise suppressor is still necessary to cut down, to reasonable levels, the discomfort for people living near the airports.

Performance Considerations

Unfortunately, the presence of a noise suppressor in the exhaust produces thrust losses which, in general, become greater with increased acoustical effectiveness. The thrust loss is not great when the suppressor is designed for an engine with only a convergent primary nozzle. But in the case of an ejector or convergent-divergent nozzle, a considerable thrust loss of 10 to 15% can occur in the sub- and transonic stages of flight because the suppression device can cause the jet to attach to the

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